CROSSED PRODUCTS BY PARTIAL ACTIONS OF INVERSE SEMIGROUPS

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ABSTRACT. In this work, for a given inverse semigroup we will define the crossed product of an inverse semigroup by a partial action. Also, we will associate to an inverse semigroup G an inverse semigroup S_G, and we will prove that there is a correspondence between the covariant representation of G and covariant representation of S_G. Finally, we will explore a connection between crossed products of an inverse semigroup actions and crossed products by partial actions of inverse semigroups.

1. INTRODUCTION

The theory of C^*-crossed product by group partial actions and inverse semigroup actions are very well developed [2] [4]. In this paper, we show that the theory of crossed products by actions of inverse semigroups can be generalized to partial actions of inverse semigroups.

In section 2 we define a partial action of an inverse semigroup as a partial homomorphism from the inverse semigroup into a symmetric inverse semigroup on some set. We will refer the reader to [1] for an extensive treatment of partial actions of inverse semigroups. In section 2, we define the crossed products by partial actions of inverse semigroups.

It turns out that there is a close connection between crossed products by partial actions of inverse semigroups and crossed products by inverse semigroups actions. In section 4, we will show that every crossed products by partial action of an inverse semigroup is isomorphic to a crossed product by an inverse semigroup action.

2. PARTIAL ACTIONS OF INVERSE SEMIGROUPS AND COVARIANT REPRESENTATIONS

We will assume that throughout this work G is a unital inverse semigroup with unit element e and $\mathcal{A}$ is a C^*-algebra.

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We recall from [1] that a partial action of an inverse semigroup S on a set X is a partial homomorphism $\alpha : S \mapsto \mathtt{I}(X)$, that is, for each $s, t \in S$

$$\alpha(s^*)\alpha(s)\alpha(t) = \alpha(s^*)\alpha(st), \quad \alpha(s)\alpha(t)\alpha(t^*) = \alpha(st)\alpha(t^*),$$

where $\mathtt{I}(X)$ denotes the inverse semigroup of all partial bijections between subsets of X. But we use [1, Proposition 3.4] to give a definition of a partial action of an inverse semigroup.

Definition 2.1. *Suppose that S is an inverse semigroup and X is a set. By a partial action of S on X a map we mean $\alpha : S \mapsto \mathtt{I}(X)$ satisfied the following conditions:*

(i) $\alpha_s^{-1} = \alpha_{s^*}$,
(ii) $\alpha_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$ *for all* $s, t \in S$ *(where* X_s *denotes the rang of* α_s *for each* $s \in S$*),*
(iii) $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$ *for all* $x \in X_{t^*} \cap X_{t^*s^*}$.

To define a partial action α of an inverse semigroup S on an associative $\mathcal{K}$-algebra $\mathcal{A}$, we suppose in Definition 2.1 that each X_s $(s \in S)$ is an ideal of $\mathcal{A}$ and that every map $\alpha_s : X_{s^*} \mapsto X_s$ is an algebra isomorphism. Furthermore, if the inverse semigroup S is unital with unit e, we shall suppose that $X_e = \mathcal{A}$. The next Proposition shows that for such a partial action α we have α_e is the identity map on $\mathcal{A}$.

Proposition 2.2. *If α is a partial action of G on a C^*-algebra $\mathcal{A}$ then α_e is the identity map ℓ on $\mathcal{A}$.*

Proof. By definition of partial action, α_e is an invertible map on it's domain, $D_e = \mathcal{A}$. Now,

$$\ell = \alpha_e\alpha_e^{-1} = \alpha_e\alpha_{e^*} = \alpha_e\alpha_e = \alpha_e.$$

Note that we have used part (3) of Definition 2.1 in the fourth equality above. □

The following Lemma will be used in the proof of Theorem 2.4

Lemma 2.3. *If α is a partial action of G on a C^*-algebra $\mathcal{A}$, then for all $t, s_1, ..., s_n \in G$*

$$\alpha_t(D_{t^*}D_{s_1}...D_{s_n}) = D_tD_{ts_1}...D_{ts_n}.$$

Proof. For $t, s_1, ..., s_n \in G$ we have

$$\begin{aligned}
\alpha_t(D_{t^*}D_{s_1}...D_{s_n}) &= \alpha_t(D_{t^*} \cap D_{s_1} \cap ... \cap D_{t^*} \cap D_{s_n}) \\
&= \alpha_t(D_{t^*} \cap D_{s_1}) \cap ... \cap \alpha_t(D_{t^*} \cap D_{s_n}) \\
&= \alpha_t(D_t \cap D_{ts_1}) \cap ... \cap \alpha_t(D_t \cap D_{ts_n}) \\
&= \alpha_t(D_t \cap D_{ts_1} \cap ... \cap D_t \cap D_{ts_n}) \\
&= \alpha_t(D_tD_{ts_1}...D_{ts_n})
\end{aligned}$$

□

Theorem 2.4. *If α is a partial action of G on a C^*-algebra $\mathcal{A}$, then for $s_1, ..., s_n \in G$ the partial automorphism $\alpha_{s_1}...\alpha_{s_n}$ has domain $D_{s_n^*} Ds_n^* s_{n-1}^* ... D_{s_n^* ... s_1}$ and range $D_{s_1}...D_{s_1...s_n}$.*

Proof. We will use induction to prove the statement about the domain. For $n = 1$

$$dom\alpha_{s_1} = ran\alpha_{s^*} = D_{s^*}.$$

Now,

$$\begin{aligned} dom\alpha_{s_1}...\alpha_{s_n} &= \alpha_{s_n}^{-1}(dom(\alpha_{s_1}...\alpha_{s_{n-1}}) \cap ran\alpha_{s_n}) \\ &= \alpha_{s_n^*}(D_{s_{n-1}^*}...D_{s_{n-1}^*...s_1^*} \cap D_{s_n}) \\ &= \alpha_{s_n^*}(D_{s_n} D_{s_{n-1}^*}...D_{s_{n-1}^*...s_1^*}) \\ &= D_{s_n^*} Ds_n^* s_{n-1}^* ... D_{s_n^*...s_1}. \end{aligned}$$

Note that we obtained the last equality by using Lemma 2.3. For the second statement, we have

$$\begin{aligned} ran\alpha_{s_1}...\alpha_{s_n} &= dom\alpha_{s_n^*}...\alpha_{s_1^*} \\ &= D_{s_1}...D_{s_1...s_n} \end{aligned}$$

by the first statement. □

If we consider a group G as an inverse semigroup, then the two definitions of partial actions as a group and as an inverse semigroup are the same. This fact motivates us to define a covariant representation of a partial action of an inverse semigroup.

Definition 2.5. *Let α be a partial action of G on an algebra $\mathcal{A}$. A covariant representation of α is a triple $(\pi, u, \mathcal{H})$, where $\pi : \mathcal{A} \to B(\mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and for each $g \in G$, u_g is a partial isometry on $\mathcal{H}$ with initial space $\pi(D_{g^*})\mathcal{H}$ and final space $\pi(D_g)\mathcal{H}$, such that*

1. $u_g \pi(a) u_{g^*} = \pi(\alpha_g(a)) \qquad a \in D_{g^*}$,
2. $u_{st} h = u_s u_t h \qquad$ *for all* $h \in \pi(D_{t^*} D_{t^* s^*})\mathcal{H}$,
3. $u_{s^*} = u_s^*$.

Notice that by the *Cohen-Hewitt factorization* Theorem $\pi(D_g)\mathcal{H}$ is a closed subspace of $\mathcal{H}$ and so the notions of initial and final spaces make sense.

Now, we show that $u_e = 1_{\mathcal{H}}$, where e denotes the unit of G. Since $D_e = \mathcal{A}$, by (2) of Definition 2.5 for all $h \in \pi(\mathcal{A})\mathcal{H} = \mathcal{H}$ we have that

$$u_e h = u_{ee} h = u_e u_e h.$$

Since u_e is one to one on $\pi(\mathcal{A})\mathcal{H} = \mathcal{H}$, we have $u_e h = h$ for all $h \in \mathcal{H}$ as we claimed.

Definition 2.6. *Let α be a partial action of G on a C^*-algebra $\mathcal{A}$. For $s \in G$, let ρ_s denote the central projection of $\mathcal{A}^{**}$ which is the identity of D_s^{**}.*

Let $(\pi, u, \mathcal{H})$ be a covariant representation of $(\mathcal{A}, G, \alpha)$. Since π is a non-degenerate representation of $\mathcal{A}$, π can be extended to a normal morphism of $\mathcal{A}^{**}$ onto $\pi(\mathcal{A})''$. We will denote this extension also by π. Note that $\pi(D_{s_1}...D_{s_n})\mathcal{H} = \pi(\rho_{s_1}\ldots\rho_{s_n})\mathcal{H}$ for all $s_1, \ldots, s_n \in G$, and $u_s u_{s^*} = \pi(\rho_s)$ for all $s \in G$.

Theorem 2.7. *Let $(\pi, u, \mathcal{H})$ be a covariant representation of $(\mathcal{A}, G, \alpha)$. Then for all $s_1, ..., s_n \in G$, $u_{s_1}...u_{s_n}$ is a partial isometry with initial space*

$$\pi(D_{s_n^*}D_{s_n^*s_{n-1}^*}...D_{s_n^*...s_1^*})\mathcal{H}$$

and final space

$$\pi(D_{s_1}...D_{s_1...s_n})\mathcal{H}.$$

Proof. Firstly, we show that $u_{s_1}...u_{s_n}u_{s_n}^*...u_{s_1}^* = \pi(\rho_{s_1}...\rho_{s_1...s_n})$. For $n = 1$ we have proved that $u_{s_1}u_{s_1}^* = \pi(\rho_{s_1})$. Now,

$$\begin{aligned} u_{s_1}...u_{s_n}u_{s_n}^*...u_{s_1}^* &= u_{s_1}u_{s_1}^*u_{s_1}\pi(\rho s_2...\rho_{s_2...s_n})ua_{s_1}^* \\ &= u_{s_1}u_{s_1^*}u_{s_1}\pi(\rho_{s_2}...\rho_{s_2...s_n})u_{s_1}^* \\ &= u_{s_1}\pi(\rho_{s_1^*})\pi(\rho_{s_2}...\rho_{s_2...s_n})u_{s_1}^* \\ &= u_{s_1}\pi(\rho_{s_1^*}\rho_{s_2}...\rho_{s_2...s_n})u_{s_1^*} \\ &= \pi(\alpha_{s_1}(\rho_{s_1^*}\rho_{s_2}...\rho_{s_2...s_n})) \\ &= \pi(\rho_{s_1}\rho_{s_1s_2}...\rho_{s_1s_2...s_n}), \end{aligned}$$

so, $u_{s_1}...u_{s_n}u_{s_n}^*...u_{s_1}^*$ is a projection since $\rho_{s_1}, ..., \rho_{s_1...s_n}$ are commute. Finally, the initial space of $u_{s_1}...u_{s_n}$ is equal to

$$\begin{aligned} u_{s_1}...u_{s_n}u_{s_n}^*...u_{s_1}^*\mathcal{H} &= \pi(\rho_{s_1}...\rho_{s_1...s_n})\mathcal{H} \\ &= \pi(D_{s_n^*}D_{s_n^*s_{n-1}^*}...D_{s_n^*...s_1^*})\mathcal{H}. \end{aligned}$$

Similarly, we can prove that the final space of $u_{s_1}...u_{s_n}$ is equal to

$$\pi(D_{s_1}...D_{s_1...s_n})\mathcal{H}.$$

□

Corollary 2.8. *If $(\pi, u, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, G, \alpha)$, then*

$$u_{s_1...s_n}h = u_{s_1}...u_{s_n}h \quad \textit{for all } h \in \pi(D_{s_n^*}...D_{s_n^*...s_1^*})\mathcal{H},$$

and

$$\pi(a)u_{s_1...s_n} = \pi(a)u_{s_1}...u_{s_n} \ \textit{for all } a \in D_{s_1}D_{s_1s_2}...D_{s_1...s_n}.$$

Proof. For $n = 2$, if $h \in \pi(D_{s_2^*}D_{s_2^*s_1^*})$ then by Definition 2.5 part (2) we have $u_{s_1s_2}h = u_{s_1}u_{s_2}$. For $h \in \pi(D_{s_n^*}...D_{s_n^*...s_1^*})\mathcal{H}$ we have $u_{s_1...s_n}h = u_{s_1...s_{n-1}}u_{s_n}h$ by Definition 2.5 part (2). Now, since

$$\begin{aligned} u_{s_n}h \in u_{s_n}\pi(\rho_{s_n^*}...\rho_{s_n^*...s_1^*})\mathcal{H} &= \pi(\rho_{s_n^*}\rho_{s_n^*s_{n-1}^*}...\rho_{s_{n-1}^*...s_1^*})\mathcal{H} \\ &\subseteq \pi(\rho_{s_{n-1}^*}...\rho_{s_{n-1}^*...s_1^*})\mathcal{H} \\ &= \pi(D_{s_{n-1}^*}...D_{s_{n-1}^*...s_1^*})\mathcal{H}, \end{aligned}$$

by induction hypothesis we have $u_{s_1...s_{n-1}}u_{s_n}h = u_{s_1}...u_{s_{n-1}}u_{s_n}h$. By the first statement, we have

$$u_{s_n^* s_{n-1}^* ... s_1^*}\pi(a^*) = u_{s_n^*}...u_{s_1^*}\pi(a^*) \tag{1}$$

since $\pi(a^*) \in \pi(D_{s_1}...D_{s_1...s_n})$ for $a \in D_{s_1}...D_{s_1...s_n}$. Taking the conjugate, we have $\pi(a)u_{s_1...s_n} = \pi(a)u_{s_1}...u_{s_n}$. □

Corollary 2.9. *If $(\pi, u, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, G, \alpha)$, then $S = \{u_{s_1}...u_{s_n} \ : \ s_1, ..., s_n \in G\}$ is a unital inverse semigroup of partial isometries of $\mathcal{H}$.*

Now, we are able to define an inverse semigroup associated to a covariant representation of a unital inverse semigroup G.

Proposition 2.10. *Let α be a partial action of an inverse semigroup G on the C^*-algebra $\mathcal{A}$, and let $(\pi, u.\mathcal{H})$ be a covariant representation of α. Let $S_G = \{(\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n}) \ : \ g_1, ..., g_n \in G\}$. Then S_G is a unital inverse semigroup with coordinate wise multiplication. For $s = (\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n}) \in S_G$ let*

$$E_s = D_{g_1}...D_{g_1...g_n},$$

and

$$\beta_s = \alpha_{g_1}...\alpha_{g_n} : E_{s^*} \to E_s.$$

Then β is an action of S_G on $\mathcal{A}$.

Proof. Let $s = (\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n}), t = (\alpha_{h_1}...\alpha_{h_n}, u_{h_1}...u_{h_n})$, then $st = (\alpha_{g_1}...\alpha_{g_n}\alpha_{h_1}...\alpha_{h_n}, u_{g_1}...u_{g_n}u_{h_1}...u_{h_n}) \in S_G$, and the unit of S_G is (α_e, u_e). Obviously E_s is a closed ideal of $\mathcal{A}$ and β_s is an isomorphism. Now, we define the domain of β_s.

$$\begin{aligned} dom(\alpha_{g_1}...\alpha_{g_n}) &= \alpha_{g_n^*}(dom(\alpha_{g_1}...\alpha_{g_{n-1}})D_{g_n}) \\ &= \alpha_{g_n^*}(\alpha_{g_{n-1}^*}(dom(\alpha_{g_1}...\alpha_{g_{n-2}})D_{g_{n-1}})D_{g_n})) \\ &\vdots \\ &= D_{g_n^*}...D_{g_n^*...g_1^*} = E_{s^*}. \end{aligned}$$

Now, let us show that $ran\beta_s = E_s$. To do this, we will use induction. For $n = 2$,

$$\begin{aligned} ran\beta_s &= ran\alpha_{g_1}\alpha_{g_2} \\ &= \alpha_{g_1}(D_{g_2}D_{g_1^*}) \\ &= D_{g_1}D_{g_1g_2} = E_s. \end{aligned}$$

On the other hand, for $s=(\alpha_{g_1}...\alpha_{g_n},u_{g_1}...u_{g_n})$,

$$\begin{aligned} ran\beta_s &= ran\alpha_{g_1}...\alpha_{g_n} \\ &= \alpha_{g_1}(ran(\alpha_{g_2}...\alpha_{g_n})D_{g_1^*}) \\ &= \alpha_{g_1}(D_{g_2}D_{g_2g_3}...D_{g_2...g_n}D_{g_1^*}) \\ &= D_{g_1}D_{g_1g_2}...D_{g_1...g_n}=E_s. \end{aligned}$$

So, $\beta_s : E_{s^*} \to E_s$ is an isomorphism, and clearly for $s,t \in S$ we have $\beta_s\beta_t=\beta_{st}$. □

The following Proposition shows that there exists a relation between covariant representation of $(\mathcal{A},S_G,\beta)$ and covariant representation of $(\mathcal{A},G,\alpha)$.

Proposition 2.11. *keeping the notation of Proposition 2.10, define $\nu : S_G \to B(\mathcal{H})$ by $\nu_s=u_{g_1}...u_{g_n}$, where $s=(\alpha_{g_1}...\alpha_{g_n},u_{g_1}...u_{g_n})$. Then $(\pi,\nu,\mathcal{H})$ is a covariant representation of $(\mathcal{A},S_G,\beta)$. Conversely, if $(\rho,z,\mathcal{K})$ is a covariant representation of $(\mathcal{A},S_G,\beta)$, then the function $\omega : G \to B(\mathcal{K})$ defined by $\omega_g=z(\alpha_g,u_g)$ gives a covariant representation $(\rho,\omega,\mathcal{K})$ of $(\mathcal{A},G,\alpha)$.*

Proof. Let $s=(\alpha_{g_1}...\alpha_{g_n},u_{g_1}...u_{g_n}) \in S$. By Theorem 2.7, $\nu_s=u_{g_1}...u_{g_n}$ is a partial isometry with initial space $\pi(E_{s^*})\mathcal{H}$ and final space $\pi(E_s)\mathcal{H}$. Obviously, ν is multiplicative. Let $a \in E_{s^*}=D_{g_n^*}...D_{g_n^*...g_1^*}$, then

$$\begin{aligned} \nu_s\pi(a)\nu_{s^*} &= u_{g_1}...u_{g_n}\pi(a)u_{g_n^*}...u_{g_1^*} \\ &= u_{g_1}...u_{g_{n-1}}\pi(\alpha_{g_n}(a))u_{g_{n-1}^*}...u_{g_1^*} \\ &\ \vdots \\ &= \pi(\alpha_{g_1}...\alpha_{g_n}(a))=\pi(\beta_s(a)). \end{aligned}$$

Conversely, suppose that $(\rho,z,\mathcal{K})$ is a covariant representation of $(\mathcal{A},S_G,\beta)$. We want to show that $(\rho,\omega,\mathcal{K})$ is a covariant representation of $(\mathcal{A},G,\alpha)$. By the definition of ω_g, $g \in G$, ω_g is a partial isometry with initial space $\pi(D_{g^*})\mathcal{K}$ and final space $\pi(D_g)\mathcal{K}$. For $g_1,g_2 \in G$, put $s=(\alpha_{g_1g_2},u_{g_1g_2})$, $s_1=(\alpha_{g_1},u_{g_1})$, and $s_2=(\alpha_{g_2},u_{g_2})$. By the definition of partial action, if $x \in D_{g_2^*}D_{g_1^*}$ then $\alpha_{g_1}\alpha_{g_2}(x)=\alpha_{g_1g_2}(x)$. But,

$$ran\alpha_{g_2^*}\alpha_{g_1^*}=\alpha_{g_2^*}(D_{g_1^*}D_{g_2})=D_{g_2^*}D_{g_2^*g_1^*}.$$

Consequently,

$$\alpha_{g_1g_2}(\alpha_{g_1}\alpha_{g_2})^*=\alpha_{g_1g_2}\alpha_{g_2^*}\alpha_{g_1^*}=\alpha_{g_1}\alpha_{g_2}(\alpha_{g_1}\alpha_{g_2}). \tag{2}$$

By Definition 2.5 part (2), $u_{g_1g_2}=u_{g_1}u_{g_2}$ on $\pi(D_{g_2^*}D_{g_2^*g_1^*})\mathcal{H}$. On the other hand, by Theorem 2.7 final space of $(u_{g_1}u_{g_2})^*$ is $\pi(D_{g_2^*}D_{g_2^*g_1^*})\mathcal{H}$. Thus

$$u_{g_1g_2}(u_{g_1}u_{g_2})^*=u_{g_1}u_{g_2}(u_{g_1}u_{g_2})^*. \tag{3}$$

Hence

$$\begin{aligned} s(s_1s_2)^* &= (\alpha_{g_1g_2}, u_{g_1g_2})[(\alpha_{g_1}, u_{g_1})(\alpha_{g_2}, u_{g_2}))]^* \\ &= (\alpha_{g_1}\alpha_{g_2}(\alpha_{g_1}\alpha_{g_2})^*, u_{g_1}u_{g_2}(u_{g_1}u_{g_2})^*) \\ &= s_1s_2(s_1s_2)^*, \end{aligned} \tag{4}$$

note that we have used equations 2 and 3 in the second equality above. So,

$$\begin{aligned} z_s z_{(s_1s_2)^*} &= z_{s(s_1s_2)^*} \\ &= z_{s_1s_2(s_1s_2)^*} \\ &= z_{s_1s_2} z_{(s_1s_2)^*} \end{aligned} \tag{5}$$

by equality 4. Now, for $h \in \rho(D_{g_2^*}D_{g_2^*g_1^*})\mathcal{K}$

$$\begin{aligned} z_s h &= z_{s_1s_2} h \\ &= z_{s_1} z_{s_2} h \end{aligned}$$

note that we have used 5 and the fact that $\rho(D_{g_2^*}D_{g_2^*g_1^*})\mathcal{K}$ is the final space of $z_{(s_1s_2)^*}$ in the first equality. Hence $\omega_{g_1g_2} = \omega_{g_1}\omega_{g_2}$ on $\rho(D_{g_2^*}D_{g_2^*g_1^*})\mathcal{K}$. For $g \in G$,

$$\omega_{g^*} = z_{(\alpha_{g^*}, u_{g^*})} = z_{s^*} = z_s^* = \omega_g^*.$$

Consequently, $(\mathcal{A}, G, \omega)$ is a covariant representation of $(\mathcal{A}, G, \alpha)$. □

3. Crossed Products

Mc Calanahan defines the partial crossed product $\mathcal{A} \ltimes_\alpha G$ of the C^*-algebra $\mathcal{A}$ and the group G by the partial action α as the enveloping C^*-algebra of $L = \{x \in \ell^1(G, \mathcal{A}) \ : \ x(g) \in D_g\}$ with the multiplication and involution

$$(x * y)(g) = \sum_{h \in G} \alpha_h[\alpha_{h^{-1}}(x(h))y(h^{-1}g)],$$

and

$$x^*(g) = \alpha_g(x(g^{-1})^*).$$

He shows that there is a bijective correspondence $(\pi, u, \mathcal{H}) \leftrightarrow (\pi \times u, \mathcal{H})$ between covariant representations of $(\mathcal{A}, G, \alpha)$ and non-degenerate representations of $\mathcal{A} \ltimes_\alpha G$, where $\pi \times u$ defined by $x \mapsto \sum_{g \in G} \pi(x(g))u_g$. We are going to follow his footsteps constructing the crossed product of a C^*-algebra and a unital inverse semigroup by a partial action.

Let α be a partial action of the unital inverse semigroup G on the C^*-algebra $\mathcal{A}$. Consider the subset $L = \{x \in \ell^1(G, \mathcal{A}) \ : \ x(g) \in E_g\}$ of $\ell^1(G, \mathcal{A})$ with the multiplication and involution as follows:

$$\begin{aligned} (x * y)(g) &= \sum_{hk=g} \beta_h(\beta_{h^*}(x(h))y(k)), \\ x^*(g) &= \beta_s(x(g^*)^*). \end{aligned}$$

Notice that by Definition 2.1 part (ii), $(x*y)(g)\in E_g$. It is easy to see that for $x,y\in L$ we have $x*y, x^*\in L$, and

$$\|x*y\|\leq\|x\|\|y\|,$$

and

$$\|x^*\|=\|x\|.$$

Obviously, L is a closed subset of $\ell^1(G,\mathcal{A})$, so, L is a Banach space. Easily one can shows that

(i) $(x+y)^*=x^*+y^*$,
(ii) $(ax)^*=\bar{a}x^*$,
(iii) $(x*y)^*=y^**x^*$.

Proposition 3.1. *L is a Banach *-algebra.*

Proof. By the argument above, L is a Banach space closed under multiplication and involution. To show that L is a Banach *-algebra, it is enough check the associativity of multiplication. It suffices to show this for $x=a_r\delta_r, y=a_s\delta_s$, and $a_t\delta_t$. Let $\{u_\lambda\}$ be an approximate identity for E_{s^*}, then

$$\begin{aligned}
(a_r\delta_r*a_s\delta_s)*a_t\delta_t &= \beta_r(\beta_{r^*}(a_r)a_s)\delta_{rs}*a_t\delta_t\\
&= \beta_{rs}(\beta_{s^*r^*}(\beta_r(\beta_{r^*}(a_r)a_s))a_t)\delta_{rst}\\
&= \beta_{rs}(\beta_{s^*}\beta_{r^*}(\beta_r(\beta_{r^*}(a_r)a_s))a_t)\delta_{rst}\\
&= \beta_{rs}(\beta_{s^*}(\beta_{r^*}(a_r)a_s)a_t)\delta_{rst}\\
&= \lim_\lambda \beta_{rs}(\beta_{s^*}(\beta_{r^*}(a_r)a_s)u_\lambda a_t)\delta_{rst}\\
&= \lim_\lambda \beta_r\beta s(\beta_{s^*}(\beta_{r^*}(a_r)a_s)a_t)\delta_{rst}\\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)a_s\beta_s(u_\lambda a_t))\delta_{rst}\\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)u_\lambda a_t))\delta_{rst}\\
&= \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)a_t))\delta_{rst}\\
&= a_r\delta_r*(\beta_s(\beta_{s^*}(a_s)a_t)))\delta_{st}\\
&= a_r\delta_r*(a_s\delta_s*a_t\delta_t).
\end{aligned}$$

□

Note that authors in [5] prove the associativity of L in a general case, where $\mathcal{A}$ is just an algebra.

Definition 3.2. *If $(\pi,\nu.\mathcal{H})$ is a covariant representation of $(\mathcal{A},S,\beta)$, then define $\pi\times\nu : L\mapsto B(\mathcal{H})$ by $(\pi\times\nu)(x)=\sum_{s\in S}\pi(x(s))\nu_s$.*

Proposition 3.3. *$\pi\times\nu$ is a non-degenerate representation of L.*

Proof. clearly $\pi \times \nu$ is a linear map from L into $B(\mathcal{H})$. As for multiplicativity, it suffices to verify this for elements of the form $a_s\delta_s$. For such elements, we have

$$\begin{aligned}\pi \times \nu(a_s\delta_s * a_t\delta_t) &= \pi \times \nu(\beta_s(\beta_{s^*}(a_s)a_t)\delta_{st}) \\ &= \pi(\beta_s(\beta_{s^*}(a_s)a_t)\nu_{st}.\end{aligned}$$

We also have

$$\begin{aligned}\pi \times \nu(a_s\delta_s)\pi \times \nu(a_t\delta_t) &= \pi(a_s)\nu_s\pi(a_t)\nu_t \\ &= \nu_s\nu_{s^*}\pi(a_s)\nu_s\pi(a_t)\nu_t \\ &= \nu_s\pi(\beta_{s^*}(a_s))\pi(a_t)\nu_t \\ &= \nu_s\pi(\beta_{s^*}(a_s)a_t)\nu_t \\ &= \nu_s\pi(\beta_{s^*}(a_s)a_t)\nu_{s^*}\nu_s\nu_t \\ &= \pi(\beta_s(\beta_{s^*}(a_s)a_t))\nu_s\nu_t\end{aligned}$$

We have used the fact that $\nu_s\nu_{s^*}\pi(a_s) = \pi(a_s)\nu_{s^*}\nu_s = \pi(a_s)$ for $a \in E_s$ in the second and fifth equalities above. Since $\beta_s(\beta_{s^*}(a_s)a_t)$ is in $\beta_s(E_{s^*}E_t) = E_sE_{st}$, it follows from Definition 2.5 that

$$\pi(\beta_s(\beta_{s^*}(a_s)a_t))\nu_s\nu_t = \pi(\beta_s(\beta_{s^*}(a_s)a_t))\nu_{st},$$

so, the multiplicativity of $\pi \times \nu$ follows. The following computations verify that $\pi \times \nu$ preserves the $*$-operation.

$$\begin{aligned}\pi \times \nu((a_s\delta_s)^*) &= \pi \times \nu(\beta_{s^*}(a_s^*)\delta_{s^*}) \\ &= \pi(\beta_{s^*}(a_s^*))\nu_{s^*} \\ &= \nu_{s^*}\pi(a_s^*)\nu_s\nu_{s^*} \\ &= \nu_{s^*}\pi(a_s^*) \\ &= (\pi(a_s)\nu_s)^* = (\pi \times \nu(a_s\delta_s))^*.\end{aligned}$$

If $\{u_\lambda\}$ is a bounded approximate identity for $\mathcal{A}$, then $\{u_\lambda\delta_e\}$ is a bounded approximate identity for L since for $a \in E_s$ we have

$$\lim_\lambda u_\lambda\delta_e * a\delta_s = \lim_\lambda u_\lambda a\delta_s = a\delta_s,$$

and

$$\lim_\lambda a\delta_s * u_\lambda\delta_e = \lim_\lambda \beta_s(\beta_{s^*}(a)u_\lambda)\delta_s = a\delta_s.$$

Since π is a non-degenerate representation, $\pi \times \nu(u_\lambda\delta_e) = \pi(u_\lambda)$ converges strongly to $1_{B(\mathcal{H})}$ and so $\pi \times \nu$ is non-degenerate. □

Definition 3.4. *Let $\mathcal{A}$ be a C^*-algebra and β be a partial action of the unital inverse semigroup G on $\mathcal{A}$. Define a seminorm $\|.\|_1$ on L by*

$$\|x\|_1 = \sup\{\|\pi \times \nu(x)\| \ : \ (\pi,\nu) \text{is a covariant representation of } (\mathcal{A},G,\beta)\},$$

and let $N = \{x \in L \ : \ \|x\|_1 = 0\}$

The crossed product $\mathcal{A} \ltimes_\beta G$ is the C^*-algebra obtained by completing the quotient $\frac{L}{N}$ with respect to $\|.\|_1$.

Lemma 3.5. *If* $s \leq t$ *in* G*, then* $\Phi(a\delta_s) = \Phi(a\delta_t)$ *for all* $a \in E_s$*, where* Φ *is the quotient map of* L *onto* $\frac{L}{N}$*.*

Proof. Notice that since $s \leq t$ there is an idempotent f in G such that $s = ft$, and we have $E_s \subseteq E_t$ by [1, Proposition 3.8], so, $a \in E_t$. If (π, ν) is a covariant representation of $(\mathcal{A}, G, \beta)$, then

$$\begin{aligned} \pi \times \nu(a\delta_s - a\delta_t) &= \pi(a)\nu_s - \pi(a)\nu_t \\ &= \pi(a)\nu_{ft} - \pi(a)\nu_t \\ &= \pi(a)\nu_f\nu_t - \pi(a)\nu_t. \end{aligned}$$

We have used the fact that for $a \in E_s$ $\pi(a)\nu_{ft} = \pi(a)\nu_f\nu_t$ in the the third equality. Since f is an idempotent, ν_f is identity on $\pi(E_f)\mathcal{H}$. Now for $h \in \mathcal{H}$ if $\nu_t(h) \in \pi(E_f)\mathcal{H}$, then

$$\pi(a)\nu_f\nu_t(h) - \pi(a)\nu_t(h) = 0.$$

If $\nu_t(h) \in (\pi(E_f)\mathcal{H})^\perp = Ker\ \nu_f$, then

$$\pi(a)\nu_f\nu_t(h) = 0.$$

On the other hand, $\pi(a)\nu_t(h) = 0$ because if $k \in \mathcal{H}$ then

$$< \pi(a)\nu_t(h), k >=< \nu_t(h), \pi(a^*)k >= 0$$

since $a^* \in E_s = E_{ft} \subseteq E_f$. Hence, $\Phi(a\delta_s - a\delta_t) = 0$. Note that the fact that $E_{ft} \subseteq E_f$ follows from [1, Corollary 2.21] and the fact that

$$E_{ft} = ran\beta_{ft} = \beta_f\beta_t = \beta_f(E_tE_f) = E_{ft} \cap E_f.$$

$\square$

Corollary 3.6. *If* G *is a semilattice, then* $\mathcal{A} \ltimes_\beta G$ *is isomorphic to* $\mathcal{A}$*.*

Proof. Let e be the identity element of G, then $g \leq e$ for each $g \in G$. Define $\psi_1 : \mathcal{A} \to \mathcal{A} \ltimes_\beta G$ by $a \mapsto \Phi(a\delta_e)$. Obviously ψ_1 is a $*$-homomorphism. Now, define $\psi_2 : \frac{L}{N} \to \mathcal{A}$ by $\Phi(a\delta_g) \mapsto a$. Now, we will show that ψ_2 is well-defined. If $\Phi(a\delta_e) = \Phi(b\delta_e)$, then for each covariant representation $(\pi, \nu, \mathcal{H})$ we have

$$\pi \times \nu(a\delta_e - b\delta_e) = \pi(a - b) = 0,$$

so, $a - b = 0$ since $\mathcal{A}$ has a universal representation. This shows that ψ_1 is well-defined since $\Phi(a\delta_g) = \Phi(a\delta_e)$ for each $g \in G$. Clearly, ψ_2 is a $*$-homomorphism that can be extended to $\mathcal{A} \ltimes_\beta G$. Finally, it is easy to see that $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ are identity maps on $\mathcal{A} \ltimes_\beta G$ and $\mathcal{A}$ respectively. $\square$

Proposition 3.7. *Let $(\Pi, \mathcal{H})$ be a non-degenerate representation of $\mathcal{A} \ltimes_\beta G$. Define a representation π of $\mathcal{A}$ on $\mathcal{H}$ and a map $\nu : S \to B(\mathcal{H})$ by*

$$\pi(a) = \Pi(a\delta_e), \qquad \nu_s = \lim_\lambda \Pi(u_\lambda \delta_s)\rho_{s^*},$$

where $\{u_\lambda\}$ is an approximate identity of E_s, limit is the strong limit, and ρ_{s^*} is the orthogonal projection onto $\pi(E_{s^*})\mathcal{H}$. Then $(\pi, \nu, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, G, \beta)$.

Proof. Clearly π is a representation of $\mathcal{A}$ on $\mathcal{H}$. Now, let $\{u_\lambda\}$ be an approximate identity for E_s, and let $h \in \mathcal{H}$. We will consider two cases:
If $h \in \pi(E_{s^*})\mathcal{H}$: then there exist elements $a \in E_{s^*}$ and $h' \in \mathcal{H}$ such that $h = \pi(a)h'$. So,

$$\begin{aligned}
\nu_s(h) &= \lim_\lambda \Pi(u_\lambda \delta_s)(\Pi(a\delta_e)h') \\
&= \lim_\lambda \Pi(u_\lambda \delta_s * a\delta_e)h' \\
&= \lim_\lambda \Pi(\beta_s(\beta_{s^*}(u_\lambda)a)\delta_s)h' \\
&= \Pi(\beta_s(a)\delta_s)h'.
\end{aligned}$$

If $h \in (\pi(E_{s^*})\mathcal{H})^\perp$: by the definition we have

$$\nu_s = \lim_\lambda \Pi(u_\lambda \delta_s)\rho_{s^*} h = 0.$$

This show that ν_s is independent of the choice of approximate identity of E_s, so ν is well-defined. Now, we want to show that $\nu_s^* = \nu_{s^*}$ for $s \in S$. First we remark that for $a_s \in E_s$ we have $\Pi(a_s\delta_s)\rho_{s^*} = \rho_s \Pi(a_s\delta_s)$. Let $\{u_\lambda\}$ be an approximate identity for E_s. It follows that

$$\begin{aligned}
(\nu_s)^* &= \lim_\lambda (\Pi(u_\lambda \delta_s)\rho_{s^*})^* \\
&= \lim_\lambda \rho_{s^*} \Pi(\beta_{s^*}(u_\lambda)\delta_{s^*}) \\
&= \lim_\lambda \Pi(\beta_{s^*}(u_\lambda)\delta_{s^*})\rho_s \\
&= \nu_{s^*}
\end{aligned}$$

since $\{\beta_{s^*}(u_\lambda)\}$ is an approximate identity for E_{s^*}. As for the covariance condition, let $x \in E_{s^*}$ and observe that

$$\begin{aligned}
\nu_s \pi(x) \nu_{s^*} &= \lim_{\lambda,\mu} \rho_s \Pi(u_\mu \delta_s) \Pi(x\delta_e) \Pi(\beta_{s^*}(u_\lambda)\delta_{s^*})\rho_s \\
&= \lim_{\mu,\lambda} \rho_s \Pi(u_\mu \delta_s * x\delta_e * \beta_{s^*}(u_\lambda)\delta_{s^*})\rho_s \\
&= \lim_{\mu,\lambda} \rho_s \Pi(u_\mu \beta_s(x) u_\lambda \delta_{ss^*})\rho_s \\
&= \lim_{\mu,\lambda} \rho_s \Pi(u_\mu \beta_s(x) u_\lambda \delta_e)\rho_s \\
&= \rho_s \pi(\beta_s(x))\rho_s \\
&= \pi(\beta_s(x)).
\end{aligned}$$

It should be noted that we have used the fact that $\Pi \equiv 0$ on N in the forth equality above. As for property (2) of Definition 2.5, notice that for $a_s \in E_s$ we have

$$\begin{aligned}
\Pi(a_s\delta_s) &= \lim_{\lambda} \Pi(a_s u_\lambda \delta_s) \\
&= \lim_{\lambda} \Pi(a_s \delta_e * u_\lambda \delta_s) \\
&= \pi(a_s)\rho_s \lim_{\lambda} \Pi(u_\lambda \delta_s) \\
&= \pi(a_s) \lim_{\lambda} \Pi(u_\lambda \delta_s)\rho_{s^*} \\
&= \pi(a_s)\nu_s.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Pi(a_s\delta_s)\Pi(a_t\delta_t) &= \pi(a_s)\nu_s \pi(a_t)\nu_t \\
&= \nu_s \nu_{s^*} \pi(a_s)\nu_s \pi(a_t)\nu_t \\
&= \nu_s \pi(\beta_{s^*}(a_s)a_t)\nu_t \\
&= \nu_s \pi(\beta_{s^*}(a_s)a_t)\nu_{s^*}\nu_s\nu_t \\
&= \pi(\beta_s(\beta_{s^*}(a_s)a_t))\nu_s\nu_t.
\end{aligned}$$

Because Π is multiplicative, the above expression is the same as

$$\begin{aligned}
\Pi(a_s\delta_s * a_t\delta_t) &= \Pi(\beta_s(\beta_{s^*}(a_s)a_t)\delta_{st}) \\
&= \pi(\beta_s(\beta_{s^*}(a_s)a_t))\nu_{st}.
\end{aligned}$$

Elements of the form $\beta_{s^*}(a_s)a_t$ generate $E_{s^*}E_t$. Since β_s maps $E_{s^*}E_t$ onto E_sE_{st}, it follows that elements of the form $\beta_s(\beta_{s^*}(a_s)a_t)$ generate E_sE_{st} and so property (2) of Definition 2.5 follows. Clearly, π is a non-degenerate representation of $\mathcal{A}$. Thus $(\pi, \nu, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, S, \beta)$. $\square$

Proposition 3.8. *The correspondence* $(\pi, \nu, \mathcal{H}) \leftrightarrow (\pi \times \nu, \mathcal{H})$*is a bijection between covariant representations of* $(\mathcal{A}, S, \beta)$ *and non-degenerate representations of* $\mathcal{A} \ltimes_\beta S$.

Proof. We will show that the correspondences $(\pi,\nu,\mathcal{H}) \mapsto (\pi\times\nu,\mathcal{H})$ and $(\Pi,\mathcal{H}) \mapsto (\pi,\nu,\mathcal{H})$ are inverses of each other. Let $(\pi',\nu',\mathcal{H})$ be a covariant representation of $(\mathcal{A},S,\beta)$. Let $(\pi,u,\mathcal{H})$ be a covariant representation of $(\mathcal{A},S,\beta)$ induced by $\pi'\times\nu'$. Then for $a\in\mathcal{A}$ and $s\in S$ we have

$$\pi(a)=\pi'\times\nu'(a\delta_e)=\pi'(a)$$

and

$$\begin{aligned} u_s &= \lim_\lambda \rho_s\pi'\times\nu'(\omega_\lambda\delta_s)\\ &= \lim_\lambda \rho_s\pi'(\omega_\lambda)\nu'_s\\ &= \lim_\lambda \pi'(\omega_\lambda)\nu'_s=\nu'_s. \end{aligned}$$

We have used the fact that $\rho_s\pi'(\omega_\lambda)\nu'_s=\pi'(\omega_\lambda)\nu'_s$ since ρ_s is the orthogonal projection onto $\pi'\times\nu'(E_s)\mathcal{H}=\overline{\mathsf{Span}}\{\pi'(a_s)\nu'_s \;:\; a_s\in E_s\}$. Let Π be a non-degenerate representation of $\mathcal{A}\ltimes_\beta S$ on $\mathcal{H}$. Let $(\pi,\nu,\mathcal{H})$ be a covariant representation of $(\mathcal{A},S,\beta)$ induced by Π. Then if $a_s\in E_s$ we have

$$\begin{aligned} \pi\times\nu(a_s\delta_s) &= \pi(a_s)\nu_s\\ &= \Pi(a_s\delta_e)\lim_\lambda\Pi(u_\lambda\delta_s)\rho_{s^*}\\ &= \Pi(a_s\delta_e)\rho_s\lim_\lambda\Pi(u_\lambda\delta_s)\\ &= \Pi(a_s\delta_e)\lim_\lambda\Pi(u_\lambda\delta_s)\\ &= \lim_\lambda\Pi(a_su_\lambda\delta_s)=\Pi(a_s\delta_s). \end{aligned}$$

Thus the correspondence is bijective. □

4. Conection Between Croossed Products

Throughout this section we will assume that G is an inverse semigroup with unit element e.

Lemma 4.1. *Let $(\mathcal{A},G,\alpha)$ and $(\mathcal{A},S_G,\beta)$ be as in Proposition 2.10. Let $(\rho,z,\mathcal{K})$ be a covariant representation of $(\mathcal{A},S_G,\beta)$, and define a covariant representation $(\rho,\omega,\mathcal{K})$ of $(\mathcal{A},G,\alpha)$ by $\omega_g=z(\alpha_g,u_g)$ as in Proposition 2.11. Then $(\rho\times z)(\mathcal{A}\ltimes_\beta S)=(\rho\times\omega)(\mathcal{A}\times_\alpha G)$.*

Proof. For $g\in G$, let $s=(\alpha_g,u_g)\in S$, then $E_s=D_g$, so, $\rho(D_g)\omega_g=\rho(E_s)z_s$. Thus,

$$\sum_{g\in G}\rho(D_g)\omega_g\subseteq\sum_{s\in S}\rho(E_s)z_s. \tag{6}$$

On the other hand, if

$$s=(\alpha_{g_1}...\alpha_{g_n},u_{g_1}...u_{g_n})\text{ and }a\in E_s=D_{g_1}D_{g_1g_2}...D_{g_1\dots g_n}$$

then by Corollary 2.8 we have

$$\begin{aligned} \rho(a)z_s &= \rho(a)z_{(\alpha_{g_1},u_{g_1})...(\alpha_{g_n},u_{g_n})} \\ &= \rho(a)z_{(\alpha_{g_1},u_{g_1})}...z_{(\alpha_{g_n},u_{g_n})} \\ &= \rho(a)\omega_{g_1}...\omega_{g_n}. \end{aligned} \tag{7}$$

Let $\Phi(\sum a_g\delta_g) \in \frac{L}{N}$. Then by 6 we have

$$\rho \times \omega(\Phi(\sum a_g\delta_g)) = \sum \rho(a_g)\omega_g \subseteq (\rho \times z)(\mathcal{A} \ltimes_\beta S),$$

so, $(\rho \times \omega)(\mathcal{A} \ltimes_\alpha G) \subseteq (\rho \times z)(\mathcal{A} \ltimes_\beta S)$. If $\Phi(\sum a_s\delta_s) \in \mathcal{A} \ltimes_\beta S$, then

$$\rho \times z(\Phi(\sum a_s\delta_s)) = \sum \rho(a_s)z_s \in \rho \times \omega(\mathcal{A} \ltimes_\alpha G)$$

by 7. □

Theorem 4.2. *Let α be a partial action of a unital inverse semigroup G on a C^*-algebra $\mathcal{A}$ such that the representation $\pi \times u$ of $\mathcal{A} \ltimes G$ is faithful. Define an inverse semigroup S_G by $S_G = \{(\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n}) : g_1, ..., g_n \in G\}$ and an action β of S_G by $\beta_s = \alpha_{g_1}...\alpha_{g_n}$ for $s = (\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n})$, as in Proposition 2.10. Then the crossed product $\mathcal{A} \ltimes_\alpha G$ and $\mathcal{A} \ltimes_\beta S$ are isomorphic.*

Proof. Let $\nu_s = u_{g_1}...u_{g_n})$ for $s = (\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n})$. We know from Proposition 2.11 $(\pi, \nu, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, S, \beta)$. If we show that $\pi \times \nu$ is a faithful representation of $\mathcal{A} \ltimes_\beta S$, then $(\pi \times \nu)^{-1} \circ \pi \times \nu$ is an isomorphism. Consider the universal representation of $\mathcal{A} \ltimes_\beta S$, which by proposition 3.8 must be in the form $\rho \times z$ for some covariant representation (ρ, z) of $(\mathcal{A}, S, \beta)$. By Proposition 2.11 the definition $\omega_g = z_{(\alpha_g,u_g)}$ gives a covariant representation $(\rho, \omega, \mathcal{K})$ of $(\mathcal{A}, G, \alpha)$ and we have $(\rho \times \omega)(\mathcal{A} \ltimes_\alpha G) = (\rho \times z)(\mathcal{A} \ltimes_\beta S)$ by Lemma 4.1. Put $\Theta(x) = (\rho \times \omega)(\pi \times u)^{-1}(x)$, thus, $\Theta \circ \pi \times u = \rho \times \omega$. We will show that $\Theta \circ (\pi \times \nu) = \rho \times z$. It suffices to check this on generators $a\delta_s$, where $s = (\alpha_{g_1}...\alpha_{g_n}, u_{g_1}...u_{g_n})$ and $a \in E_s = D_{g_1}...D_{g_1...g_n}$.

$$\begin{aligned} \Theta((\pi \times \nu)(a\delta_s)) &= \Theta(\pi(a)\nu_s) \\ &= (\rho \times \omega)(\pi \times u)^{-1}(\pi(a)\nu_s) \\ &= (\rho \times \omega)(\pi \times u)^{-1}(\pi(a)u_{g_1}...u_{g_n}) \\ &= (\rho \times \omega)(\pi \times u)^{-1}(\pi(a)u_{g_1...g_n}) \\ &= (\rho \times \omega)(a\delta_{g_1...g_n}) \\ &= \rho(a)\omega_{g_1...g_n} \\ &= \rho(a)\omega_{g_1}...\omega_{g_n} \\ &= \rho(a)z_{(\alpha_{g_1},u_{g_1})}...z_{(\alpha_{g_n},u_{g_n})} \\ &= \rho(a)z_{(\alpha_{g_1}...\alpha_{g_n},u_{g_1}...u_{g_n})} \\ &= \rho \times z(a\delta_s) \end{aligned}$$

where we have appealed to Corollary 2.8 twice more. □

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